

ベクトル解析 — 便利な公式一覧

- $\mathbf{A} = (A_x, A_y, A_z)$, $\mathbf{B} = (B_x, B_y, B_z)$, $\mathbf{C} = (C_x, C_y, C_z)$, $\mathbf{0} = (0, 0, 0)$
- $\phi = \text{const.}$ (スカラー定数)
- x, y, z 方向への単位ベクトル: $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$

■ 定義: 内積 (スカラー積)

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

■ 定義: 外積 (ベクトル積)

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (A_y B_z - A_z B_y, A_z B_x - A_x B_z, A_x B_y - A_y B_x) \\ &= (A_y B_z - A_z B_y) \mathbf{e}_x + (A_z B_x - A_x B_z) \mathbf{e}_y + (A_x B_y - A_y B_x) \mathbf{e}_z \end{aligned}$$

■ 基本公式

$$\mathbf{A} \cdot \mathbf{A} = A^2 = A_x^2 + A_y^2 + A_z^2 \qquad \mathbf{A} \times \mathbf{A} = \mathbf{0}$$

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \qquad \mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A})$$

$$\phi(\mathbf{A} \cdot \mathbf{B}) = (\phi \mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (\phi \mathbf{B}) \qquad \phi(\mathbf{A} \times \mathbf{B}) = (\phi \mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (\phi \mathbf{B})$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \cdot \mathbf{B}) + (\mathbf{A} \cdot \mathbf{C}) \qquad \mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \qquad (\mathbf{A} \times \mathbf{B})^2 = A^2 B^2 - (\mathbf{A} \cdot \mathbf{B})^2$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{A} \cdot \mathbf{B}) \mathbf{C} \qquad (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{B} \cdot \mathbf{C}) \mathbf{A}$$

■ 定義: ナブラベクトル

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z}$$

■ 微分公式

$$\nabla \cdot \mathbf{A} = \text{div} \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\begin{aligned} \nabla \times \mathbf{A} = \text{rot} \mathbf{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\ &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \mathbf{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{e}_z \end{aligned}$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \qquad \Leftrightarrow \text{div}(\mathbf{A} + \mathbf{B}) = \text{div} \mathbf{A} + \text{div} \mathbf{B}$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B} \qquad \Leftrightarrow \text{rot}(\mathbf{A} + \mathbf{B}) = \text{rot} \mathbf{A} + \text{rot} \mathbf{B}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \qquad \Leftrightarrow \text{div}(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\text{rot} \mathbf{A}) - \mathbf{A} \cdot (\text{rot} \mathbf{B})$$

三角関数 — 便利な公式一覧

■ 周期性

$$\sin(\alpha \pm 2\pi) = \sin \alpha$$

$$\cos(\alpha \pm 2\pi) = \cos \alpha$$

$$\tan(\alpha \pm \pi) = \tan \alpha$$

■ 対称性と平行移動

$$\sin(-\alpha) = -\sin \alpha$$

$$\cos(-\alpha) = \cos \alpha$$

$$\tan(-\alpha) = -\tan \alpha$$

$$\sin\left(\alpha + \frac{\pi}{2}\right) = \cos \alpha$$

$$\cos\left(\alpha + \frac{\pi}{2}\right) = -\sin \alpha$$

$$\tan\left(\alpha + \frac{\pi}{4}\right) = \frac{1 + \tan \alpha}{1 - \tan \alpha}$$

$$\sin\left(\alpha - \frac{\pi}{2}\right) = -\cos \alpha$$

$$\cos\left(\alpha - \frac{\pi}{2}\right) = \sin \alpha$$

$$\tan\left(\alpha - \frac{\pi}{4}\right) = \frac{\tan \alpha - 1}{\tan \alpha + 1}$$

$$\sin(\alpha + \pi) = -\sin \alpha$$

$$\cos(\alpha + \pi) = -\cos \alpha$$

$$\tan\left(\alpha + \frac{\pi}{2}\right) = -\frac{1}{\tan \alpha}$$

$$\sin(\alpha - \pi) = -\sin \alpha$$

$$\cos(\alpha - \pi) = -\cos \alpha$$

$$\tan\left(\alpha - \frac{\pi}{2}\right) = -\frac{1}{\tan \alpha}$$

■ 級数展開

$$\sin \alpha = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \dots$$

$$\cos \alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \dots$$

$$\tan \alpha = \alpha + \frac{1}{3}\alpha^3 + \frac{2}{15}\alpha^5 + \frac{17}{315}\alpha^7 + \dots$$

(ただし、 $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$)

■ 基本公式

$$\sin^2 \alpha + \cos^2 \alpha = 1$$

$$\Rightarrow \tan^2 \alpha + 1 = \frac{1}{\cos^2 \alpha}$$

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$\Rightarrow \tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}$$

$$\sin \alpha + \sin \beta = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\sin \alpha - \sin \beta = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos \alpha + \cos \beta = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos \alpha - \cos \beta = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$$

$$\tan \alpha \pm \tan \beta = \frac{\sin(\alpha \pm \beta)}{\cos \alpha \cos \beta}$$

$$\sin(2\alpha) = 2 \sin \alpha \cos \alpha$$

$$\begin{aligned} \cos(2\alpha) &= \cos^2 \alpha - \sin^2 \alpha \\ &= 2 \cos^2 \alpha - 1 = 1 - 2 \sin^2 \alpha \end{aligned}$$

$$\Rightarrow \tan(2\alpha) = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

$$\sin^2 \alpha = \frac{1}{2}(1 - \cos 2\alpha)$$

$$\cos^2 \alpha = \frac{1}{2}(1 + \cos 2\alpha)$$

$$\Rightarrow \tan^2 \alpha = \frac{1 - \cos 2\alpha}{1 + \cos 2\alpha}$$

$$\Rightarrow \cos(2\alpha) = \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha}$$

$$\sin(2\alpha) = \frac{2 \tan \alpha}{1 + \tan^2 \alpha}$$

$$a \sin \alpha + b \cos \alpha = \sqrt{a^2 + b^2} \sin(\alpha + \phi)$$

$$\phi = \begin{cases} \tan^{-1}(b/a) & (a \geq 0 \text{ のとき}) \\ \tan^{-1}(b/a) + \pi & (a < 0 \text{ のとき}) \end{cases}$$

■ 逆関数

$$\begin{aligned} \sin(\sin^{-1}\alpha) &= \alpha, & \sin^{-1}(\sin\alpha) &= \alpha \\ \cos(\cos^{-1}\alpha) &= \alpha, & \cos^{-1}(\cos\alpha) &= \alpha \\ \tan(\tan^{-1}\alpha) &= \alpha, & \tan^{-1}(\tan\alpha) &= \alpha \end{aligned}$$

$$\begin{aligned} \sin^{-1}(-\alpha) &= -\sin^{-1}\alpha \\ \cos^{-1}(-\alpha) &= \pi - \cos^{-1}\alpha \\ \tan^{-1}(-\alpha) &= -\tan^{-1}\alpha \end{aligned}$$

$$\sin^{-1}\alpha = \frac{\pi}{2} - \cos^{-1}\alpha$$

$$\begin{aligned} \sin(\cos^{-1}\alpha) &= \sqrt{1-\alpha^2} \\ \sin(\tan^{-1}\alpha) &= \frac{\alpha}{\sqrt{1+\alpha^2}} \end{aligned}$$

$$\begin{aligned} \cos(\sin^{-1}\alpha) &= \sqrt{1-\alpha^2} \\ \cos(\tan^{-1}\alpha) &= \frac{1}{\sqrt{1+\alpha^2}} \end{aligned}$$

$$\tan(\sin^{-1}\alpha) = \frac{\alpha}{\sqrt{1-\alpha^2}}$$

$$\tan(\cos^{-1}\alpha) = \frac{\sqrt{1-\alpha^2}}{\alpha}$$

■ 級数展開

$$\begin{aligned} \sin^{-1}\alpha &= \alpha + \frac{1}{2}\frac{\alpha^3}{3} + \frac{1}{2}\frac{3}{4}\frac{\alpha^5}{5} \\ &\quad + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{\alpha^7}{7} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{7}{8}\frac{\alpha^9}{9} + \dots \end{aligned}$$

$$\begin{aligned} \cos^{-1}\alpha &= \frac{\pi}{2} - \sin^{-1}\alpha \\ &\quad (\text{ただし、} -1 < \alpha < 1) \end{aligned}$$

$$\begin{aligned} \tan^{-1}\alpha &= \alpha - \frac{\alpha^3}{3} + \frac{\alpha^5}{5} - \frac{\alpha^7}{7} + \frac{\alpha^9}{9} - \dots \\ &\quad (\text{ただし、} -1 \leq \alpha \leq 1) \end{aligned}$$

■ ワイエルシュトラスの置換

□ $\sin\alpha, \cos\alpha$ のみの有理式 (たとえば、 $1/(1+\cos\alpha)^2$ とか) の積分では、次の変数変換が、とても有効です。この変数変換は、「ワイエルシュトラスの置換」などと呼ばれています。

$$\begin{aligned} u = \tan\left(\frac{\alpha}{2}\right) &\implies d\alpha = \frac{2}{1+u^2} du \quad \text{および} \quad \sin\alpha = \frac{2u}{1+u^2}, \quad \cos\alpha = \frac{1-u^2}{1+u^2}, \\ u &= \frac{\sin\alpha}{1+\cos\alpha}, \quad u^2 = \frac{1-\cos\alpha}{1+\cos\alpha} \end{aligned}$$

■ 微積分

$$\frac{d}{d\alpha}\sin\alpha = \cos\alpha$$

$$\frac{d}{d\alpha}\cos\alpha = -\sin\alpha$$

$$\frac{d}{d\alpha}\tan\alpha = \frac{1}{\cos^2\alpha} = 1 + \tan^2\alpha$$

$$\frac{d}{d\alpha}\sin^{-1}\alpha = \frac{1}{\sqrt{1-\alpha^2}}$$

$$\frac{d}{d\alpha}\cos^{-1}\alpha = -\frac{1}{\sqrt{1-\alpha^2}}$$

$$\frac{d}{d\alpha}\tan^{-1}\alpha = \frac{1}{1+\alpha^2}$$

□ C : 積分定数

$$\int \sin\alpha \, d\alpha = -\cos\alpha + C$$

$$\int \cos\alpha \, d\alpha = \sin\alpha + C$$

$$\int \tan\alpha \, d\alpha = -\ln|\cos\alpha| + C$$

□ C : 積分定数

$$\int \frac{1}{\sin\alpha} \, d\alpha = \frac{1}{2} \ln\left(\frac{1-\cos\alpha}{1+\cos\alpha}\right) + C$$

$$\int \frac{1}{\cos\alpha} \, d\alpha = \frac{1}{2} \ln\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right) + C$$

$$\int \frac{1}{\tan\alpha} \, d\alpha = \ln|\sin\alpha| + C$$

□ C : 積分定数

$$\int \sin^{-1}\alpha \, d\alpha = \alpha \sin^{-1}\alpha + \sqrt{1-\alpha^2} + C$$

$$\int \cos^{-1}\alpha \, d\alpha = \alpha \cos^{-1}\alpha - \sqrt{1-\alpha^2} + C$$

$$\int \tan^{-1}\alpha \, d\alpha = \alpha \tan^{-1}\alpha - \ln\sqrt{1+\alpha^2} + C$$

双曲線関数 — 便利な公式一覧

■ 定義と基本公式

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\cosh x + \sinh x = e^x$$

$$\cosh x - \sinh x = e^{-x}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh(\alpha \pm \beta) = \sinh\alpha \cosh\beta \pm \cosh\alpha \sinh\beta$$

$$\cosh(\alpha \pm \beta) = \cosh\alpha \cosh\beta \pm \sinh\alpha \sinh\beta$$

$$\tanh(\alpha \pm \beta) = \frac{\tanh\alpha \pm \tanh\beta}{1 \pm \tanh\alpha \tanh\beta}$$

$$\sinh(2\alpha) = 2\sinh\alpha \cosh\alpha$$

$$\cosh(2\alpha) = 2\cosh^2\alpha - 1 = 2\sinh^2\alpha + 1$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

$$\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \dots$$

■ 微積分

$$\frac{d}{dx}\sinh x = \cosh x$$

$$\frac{d}{dx}\cosh x = \sinh x$$

$$\frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x} = 1 - \tanh^2 x$$

■ 逆関数

$$\sinh(\sinh^{-1}x) = x, \quad \sinh^{-1}(\sinh x) = x$$

$$\cosh(\cosh^{-1}x) = x, \quad \cosh^{-1}(\cosh x) = x$$

$$\tanh(\tanh^{-1}x) = x, \quad \tanh^{-1}(\tanh x) = x$$

$$\begin{aligned} \sinh^{-1}x &= \ln(x + \sqrt{x^2+1}) \\ &= \cosh^{-1}(\sqrt{x^2+1}) \\ &= \tanh^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) \end{aligned}$$

$$\begin{aligned} \cosh^{-1}x &= \ln(x + \sqrt{x^2-1}) \\ &= \sinh^{-1}(\sqrt{x^2-1}) \\ &= \tanh^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right) \end{aligned}$$

$$\tanh^{-1}x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$

$$\sinh(\cosh^{-1}x) = \sqrt{x^2-1}$$

$$\cosh(\sinh^{-1}x) = \sqrt{x^2+1}$$

$$\tanh(\sinh^{-1}x) = \frac{x}{\sqrt{x^2+1}}$$

$$\tanh(\cosh^{-1}x) = \frac{\sqrt{x^2-1}}{x}$$

□ C : 積分定数

$$\int \sinh x \, dx = \cosh x + C$$

$$\int \cosh x \, dx = \sinh x + C$$

$$\int \tanh x \, dx = \ln(\cosh x) + C$$

